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Partition identities and Ramanujan's modular equations

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Abstract

We show that certain modular equations and theta function identities of Ramanujan imply elegant partition identities. Several of the identities are for t-cores.

Keywords: Partitions; t-Cores; Theta-functions; Modular equations

1. Introduction

H.M. Farkas and I. Kra [8] were perhaps the first mathematicians to observe that theta constant identities can be interpreted as or can be transcribed into partition identities. (Theta constant identities can be thought of as modular equations, and we use the latter terminology throughout this paper.) Perhaps the most elegant of their three partition theorems is the following result [8, Theorem 3, p. 202].

Theorem 1.1. *Let S denote the set consisting of one copy of the positive integers and one additional copy of those positive integers that are multiples of 7. Then for each positive integer k, the number of partitions of $2k$ into even elements of S is equal to the number of partitions of $2k+1$ into odd elements of S.*

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In [7], the second author observed that two results of Farkas and Kra are equivalent to two modular equations, both found by both Ramanujan, in his notebooks [13], and H. Schröter, either in his doctoral dissertation [15] from 1854 or in his papers [16,17] emanating several years later from his doctoral dissertation. It was furthermore remarked in [7] that there appear to be only five modular equations of prime degree, namely, degrees 3, 5, 7, 11, and 23, which yield partition identities of this certain type. It should be emphasized that no two of the five partition identities have exactly the same structure. We conjecture that in each case, the partition identity is unique. In other words, for each of the five partition theorems, the prime 3, 5, 7, 11, or 23 cannot be replaced by any other prime.

M.D. Hirschhorn [12] found a simple proof of the theta function identity equivalent to the theorem of Farkas and Kra [8]. Furthermore, S.O. Warnaar [19] established an extensive generalization of the partition theorem of Farkas and Kra by first proving a very general theta function identity. In fact, an equivalent formulation of Warnaar's theta function identity can be found in Ramanujan's notebooks. Thus, the first objective of this paper is to show that Warnaar's and Ramanujan's identities are really the same after a redefinition of variables.

In his notebooks [13], Ramanujan recorded over 100 modular equations, and in his lost notebook [14], Ramanujan recorded additional modular equations. Proofs for almost all of these modular equations can be found in the books of Berndt [4–6] and G.E. Andrews and B.C. Berndt [2]. Besides the five modular equations mentioned above, it is natural to ask if there are further modular equations of Ramanujan that yield partition theoretical information. The main purpose of this paper is to affirmatively answer this question and to present several new partition identities arising from Ramanujan's modular equations and/or theta function identities.

In Section 4, we use modular equations to find some identities for t-cores, and in Section 5, we derive identities for self-conjugate t-cores.

One of the identities found by Farkas and Kra arises from a modular equation of degree 3; see the papers of Warnaar [19] and Berndt [7] for further proofs. In Section 6, we establish three new theorems on partitions arising from modular equations of degree 3. One of the partition identities in [7] and another in [8] are associated with modular equations of degree 5. In Section 7, we derive three new theorems arising from modular equations of Ramanujan of degree 5.

Although we mentioned above that there appear to be only five partition theorems of the type originally found by Farkas and Kra and associated with modular equations of prime degree, there is one modular equation of Ramanujan and H. Weber [20] of composite degree of the same sort, namely, of degree 15. In Section 8, we derive the corresponding partition identity for this modular equation, and we examine two further modular equations of degree 15 of Ramanujan and derive two further partition identities. Like those identities derived in Sections 6 and 7, the latter identities are in the same spirit as the original identities of Farkas and Kra, Warnaar, and Berndt, but of a slightly different type.

Our findings offered in this paper are by no means exhaustive, and our research continues in this direction [3]. However, for the theorems proved in this paper, we feel that we have exhausted the possibilities of more theorems of this sort.

2. Definitions, preliminary results, and notation

Throughout this paper, we assume that $|q| < 1$ and use the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}\left(1 - aq^n\right).$$

We also use the standard notation

$$(a_1, a_2, \dots, a_k; q)_\infty := (a_1; q)_\infty (a_2; q)_\infty \cdots (a_k; q)_\infty. \tag{2.1}$$

For $|ab| < 1$, Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.2}$$

Jacobi's famous triple product identity [4, Entry 19, p. 35] is given by

$$f(a,b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty. \tag{2.3}$$

The three most important special cases of $f(a,b)$ are

$$\phi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \left(-q; q^2\right)_\infty^2 \left(q^2; q^2\right)_\infty, \tag{2.4}$$

$$\psi(q) := f\left(q, q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}, \tag{2.5}$$

and

$$f(-q) := f\left(-q, -q^2\right) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty, \tag{2.6}$$

where the product representation in (2.4)–(2.6) arise from (2.3). The last equality in (2.6) is also Euler's famous pentagonal number theorem. In the sequel, we use several times another famous identity of Euler, namely,

$$\left(q; q^2\right)_\infty^{-1} = (-q; q)_\infty. \tag{2.7}$$

After Ramanujan, we also define

$$\chi(q) := \left(-q; q^2\right)_\infty. \tag{2.8}$$

In the following two lemmas, we state some properties satisfied by $f(a,b)$.

Lemma 2.1. [4, Entry 30, p. 46] *We have*

$$f(a,b) + f(-a,-b) = 2f\left(a^3 b, ab^3\right) \tag{2.9}$$

and

$$f(a,b) - f(-a,-b) = 2af\left(b/a, a^5 b^3\right). \tag{2.10}$$

Lemma 2.2. [4, Entry 29, p. 45] *If* $ab = cd$, *then*

$$f(a,b)f(c,d) + f(-a,-b)f(-c,-d) = 2f(ac,bd)f(ad,bc) \tag{2.11}$$

and

$$f(a,b)f(c,d) - f(-a,-b)f(-c,-d) = 2af\left(b/c, ac^2 d\right) f\left(b/d, acd^2\right). \tag{2.12}$$

We shall also use the following lemma [4, Entries 10(iv), 10(v), p. 262].

Lemma 2.3. *We have*

$$\phi^2(q) - \phi^2\big(q^5\big) = 4qf\big(q, q^9\big)f\big(q^3, q^7\big) \tag{2.13}$$

and

$$\psi^2(q) - q\psi^2\big(q^5\big) = f\big(q, q^4\big)f\big(q^2, q^3\big). \tag{2.14}$$

We next define a modular equation as understood by Ramanujan. To that end, first define the complete elliptic integral of the first kind associated with the *modulus* k, $0 < k < 1$, by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

The complementary modulus k' is defined by $k' := \sqrt{1-k^2}$. Set $K' := K(k')$. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{2.15}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l that is implied by (2.15). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α.

If $q = \exp(-\pi K'/K)$, then one of the fundamental results in the theory of elliptic functions [4, Entry 6, p. 101] is given by

$$\phi^2(q) = \frac{2}{\pi}K(k) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \tag{2.16}$$

where ϕ is as defined in (2.4) and where ${}_2F_1(a, b; c; z)$, $|z| < 1$, denotes the ordinary or Gaussian hypergeometric series.

The identity (2.16) enables one to derive formulas for ϕ, ψ, f, and χ at different arguments in terms of α, q, and $z := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \alpha)$. In particular, Ramanujan recorded the following identities in his second notebook [13], [4, pp. 122–124].

Lemma 2.4. *We have*

$$f(q) = \frac{\sqrt{z}\{\alpha(1-\alpha)\}^{1/24}}{2^{1/6}q^{1/24}}, \tag{2.17}$$

$$f(-q) = \frac{\sqrt{z}\alpha^{1/24}(1-\alpha)^{1/6}}{2^{1/6}q^{1/24}}, \tag{2.18}$$

$$f\big(-q^2\big) = \frac{\sqrt{z}\{\alpha(1-\alpha)\}^{1/12}}{2^{1/3}q^{1/12}}, \tag{2.19}$$

$$f\big(-q^4\big) = \frac{\sqrt{z}\alpha^{1/6}(1-\alpha)^{1/24}}{2^{2/3}q^{1/6}}, \tag{2.20}$$

$$\chi(q) = 2^{1/6}\frac{q^{1/24}}{\{\alpha(1-\alpha)\}^{1/24}}, \tag{2.21}$$

$$\chi(-q) = 2^{1/6}\frac{q^{1/24}(1-\alpha)^{1/12}}{\alpha^{1/24}}, \tag{2.22}$$

$$\chi\left(-q^2\right)=2^{1/3}\frac{q^{1/12}(1-\alpha)^{1/24}}{\alpha^{1/12}}. \tag{2.23}$$

Suppose that β has degree n over α. If we replace q by q^n above, then the same evaluations hold with α replaced by β and z replaced by $z_n := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; \beta)$.

Some of our partition identities involve t-cores, which we now define. A partition λ is said to be a t-core if and only if it has no hook numbers that are multiples of t; or if and only if λ has no rim hooks that are multiples of t. If $a_t(n)$ denotes the number of partitions of n that are t-cores, then the generating function for $a_t(n)$ is given by [10, Eq. (2.1)]

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f^t(-q^t)}{f(-q)}. \tag{2.24}$$

In particular, for $t=3$ and 5,

$$\sum_{n=0}^{\infty} a_3(n)q^n = \frac{f^3(-q^3)}{f(-q)} \quad \text{and} \quad \sum_{n=0}^{\infty} a_5(n)q^n = \frac{f^5(-q^5)}{f(-q)}. \tag{2.25}$$

We also note from [10, Eq. (7.1b)] that, for t odd, the generating function for $asc_t(n)$, the number of t-cores that are self-conjugate, is given by

$$\sum_{n=0}^{\infty} asc_t(n)q^n = \frac{\chi(q)f^{(t-1)/2}(-q^{2t})}{\chi(q^t)}. \tag{2.26}$$

3. Equivalence of identities of Warnaar and Ramanujan

In [19], Warnaar proved generalizations of the partition theorems of Farkas and Kra [8,9] via the following key identity, for which Warnaar supplied three proofs. Recall the notation (2.1). Then

$$\begin{aligned}&(-c,-ac,-bc,-abc,-q/c,-q/ac,-q/bc,-q/abc;q)_\infty\\&\quad-(c,ac,bc,abc,q/c,q/ac,q/bc,q/abc;q)_\infty\\&\qquad=2c\left(-a,-b,-abc^2,-q/a,-q/b,-q/abc^2,-q,-q;q\right)_\infty.\end{aligned} \tag{3.1}$$

In this section, we show that (3.1) is equivalent to the following identity recorded by Ramanujan in his second notebook [13] and proved by C. Adiga, B.C. Berndt, S. Bhargava, and G.N. Watson [1], [4, corollary, p. 47]. If $uv=xy$, then

$$\begin{aligned}&f(u,v)f(x,y)f(un,v/n)f(xn,y/n)\\&\quad-f(-u,-v)f(-x,-y)f(-un,-v/n)f(-xn,-y/n)\\&\qquad=2uf(x/u,uy)f(y/un,uxn)f(n,uv/n)\psi(uv).\end{aligned} \tag{3.2}$$

Set

$$u=c,\qquad v=\frac{q}{c},\qquad x=\frac{q}{abc},\qquad y=abc,\quad \text{and}\quad n=a$$

in (3.2) to deduce that

$$\begin{aligned}&f(c,q/c)f(q/abc,abc)f(ca,q/ca)f(q/bc,bc)\\&\quad-f(-c,-q/c)f(-q/abc,-abc)f(-ca,-q/ca)f(-q/bc,-bc)\\&\qquad=2cf\big(q/abc^2,abc^2\big)f(b,q/b)f(a,q/a)\psi(q).\end{aligned}\tag{3.3}$$

Employing the Jacobi triple product identity (2.3) and (2.5) in (3.3), we find that

$$\begin{aligned}&(-c,-ac,-bc,-abc,-q/c,-q/ac,-q/bc,-q/abc;q)_\infty\\&\quad-(c,ac,bc,abc,q/c,q/ac,q/bc,q/abc;q)_\infty\\&\qquad=2c\big(-a,-b,-abc^2,-q/a,-q/b,-q/abc^2;q\big)_\infty\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q;q^2)_\infty}.\end{aligned}\tag{3.4}$$

Noting that $(q;q)_\infty=(q;q^2)_\infty(q^2;q^2)_\infty$, we rewrite (3.4) as

$$\begin{aligned}&(-c,-ac,-bc,-abc,-q/c,-q/ac,-q/bc,-q/abc;q)_\infty\\&\quad-(c,ac,bc,abc,q/c,q/ac,q/bc,q/abc;q)_\infty\\&\qquad=2c\big(-a,-b,-abc^2,-q/a,-q/b,-q/abc^2;q\big)_\infty\big(q;q^2\big)_\infty^{-1}\big(q;q^2\big)_\infty^{-1}.\end{aligned}\tag{3.5}$$

Employing Euler's identity (2.7) in (3.5), we readily arrive at (3.1).

4. Theorems on 3- and 5-cores

Theorem 4.1. *If $a_3(n)$ denotes the number of partitions of n that are 3-cores, then*

$$a_3(4n+1)=a_3(n).\tag{4.1}$$

Proof. We begin with the following modular equation of Ramanujan [4, Entry 5(i), p. 230]. If β has degree 3 over α, then

$$\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}-\left(\frac{\beta^3}{\alpha}\right)^{1/8}=1.\tag{4.2}$$

Multiply both sides of (4.2) by

$$\frac{z_3^{3/2}\beta^{1/8}(1-\beta)^{1/8}}{2^{1/3}\sqrt{z_1}q^{1/3}\alpha^{1/24}(1-\alpha)^{1/24}}$$

and write the resulting equation in the form

$$\begin{aligned}&\left(\frac{\sqrt{z_3}(1-\beta)^{1/6}\beta^{1/24}}{2^{1/6}q^{3/24}}\right)^3\frac{2^{1/6}q^{1/24}}{\sqrt{z_1}(1-\alpha)^{1/6}\alpha^{1/24}}\\&\quad-2q\left(\frac{\sqrt{z_3}(1-\beta)^{1/24}\beta^{1/6}}{2^{2/3}q^{3/6}}\right)^3\frac{2^{2/3}q^{1/6}}{\sqrt{z_1}(1-\alpha)^{1/24}\alpha^{1/6}}\\&\qquad=\left(\frac{\sqrt{z_3}(1-\beta)^{1/24}\beta^{1/24}}{2^{1/6}q^{3/24}}\right)^3\frac{2^{1/6}q^{1/24}}{\sqrt{z_1}(1-\alpha)^{1/24}\alpha^{1/24}}.\end{aligned}$$

We now use (2.17), (2.18), and (2.20) to write this last equality in the equivalent form

$$\frac{f^3(-q^3)}{f(-q)}-2q\frac{f^3(-q^{12})}{f(-q^4)}=\frac{f^3(q^3)}{f(q)}.\tag{4.3}$$

Employing (2.25), we can rewrite (4.3) as

$$\frac{1}{2}\left(\sum_{n=0}^{\infty} a_3(n)q^n - \sum_{n=0}^{\infty}(-1)^n a_3(n)q^n\right) = q\sum_{n=0}^{\infty} a_3(n)q^{4n}. \tag{4.4}$$

Equating the coefficients of q^{4n+1} on both sides of (4.4), we readily arrive at (4.1). □

J.A. Sellers [18] has informed us that he has derived a generalization of (4.1).

Theorem 4.2. *Let $a_5(n)$ denote the number of partitions of n that are 5-cores. Then*

$$a_5(4n+3) = a_5(2n+1) + 2a_5(n). \tag{4.5}$$

Proof. Entry 13(iii) in Chapter 19 of Ramanujan's second notebook [4, p. 280] gives the modular equation

$$\left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} - \left(\frac{\beta^5}{\alpha}\right)^{1/8} = 1 + 2^{1/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}, \tag{4.6}$$

where β has degree 5 over α. Now multiply (4.6) by

$$\frac{z_5^{5/2}(1-\beta)^{5/24}\beta^{5/24}}{\sqrt{z_1}2^{2/3}q(1-\alpha)^{1/24}\alpha^{1/24}}$$

and rearrange to obtain the identity

$$\begin{aligned}
&\left(\frac{\sqrt{z_5}(1-\beta)^{1/6}\beta^{1/24}}{2^{1/6}q^{5/24}}\right)^5 \frac{2^{1/6}q^{1/24}}{\sqrt{z_1}(1-\alpha)^{1/6}\alpha^{1/24}} \\
&\quad - 4q^3\left(\frac{\sqrt{z_5}(1-\beta)^{1/24}\beta^{1/6}}{2^{2/3}q^{5/6}}\right)^5 \frac{2^{2/3}q^{1/6}}{\sqrt{z_1}(1-\alpha)^{1/24}\alpha^{1/6}} \\
&\quad = \left(\frac{\sqrt{z_5}(1-\beta)^{1/24}\beta^{1/24}}{2^{1/6}q^{5/24}}\right)^5 \frac{2^{1/6}q^{1/24}}{\sqrt{z_1}(1-\alpha)^{1/24}\alpha^{1/24}} \\
&\qquad + 2q\left(\frac{\sqrt{z_5}(1-\beta)^{1/12}\beta^{1/12}}{2^{1/3}q^{5/12}}\right)^5 \frac{2^{1/3}q^{1/12}}{\sqrt{z_1}(1-\alpha)^{1/12}\alpha^{1/12}}.
\end{aligned}$$

Employing (2.17)–(2.20), we readily find that this last identity can be recast in the form

$$\frac{f^5(-q^5)}{f(-q)} - 4q^3\frac{f^5(-q^{20})}{f(-q^4)} = \frac{f^5(q^5)}{f(q)} + 2q\frac{f^5(-q^{10})}{f(-q^2)}. \tag{4.7}$$

Employing (2.25), we can rewrite (4.7) as

$$\frac{1}{2}\left(\sum_{n=0}^{\infty} a_5(n)q^n - \sum_{n=0}^{\infty}(-1)^n a_5(n)q^n\right) = q\sum_{n=0}^{\infty} a_5(n)q^{2n} + 2q^3\sum_{n=0}^{\infty} a_5(n)q^{4n}. \tag{4.8}$$

Equating the coefficients of q^{4n+3} on both sides of (4.8), we readily arrive at (4.5). □

5. Theorems on 3- and 5-cores that are self-conjugate

Theorem 5.1. *If $asc_3(n)$ denotes the number of 3-cores of n that are self-conjugate, then*

$$asc_3(4n+1) = asc_3(n). \tag{5.1}$$

Proof. Setting $t = 3$ in (2.26), we find that

$$\sum_{n=0}^{\infty} asc_3(n)q^n = \frac{\chi(q)f(-q^6)}{\chi(q^3)}. \tag{5.2}$$

Applying the Jacobi triple product identity (2.3) and recalling (2.6) and (2.8), we find that

$$f(q, q^5) = (-q; q^6)_\infty (-q^5; q^6)_\infty (q^6; q^6)_\infty = \frac{(-q; q^2)_\infty (q^6; q^6)_\infty}{(-q^3; q^6)_\infty} = \frac{\chi(q)f(-q^6)}{\chi(q^3)}. \tag{5.3}$$

From (5.2) and (5.3), we deduce that

$$\sum_{n=0}^{\infty} asc_3(n)q^n = f(q, q^5). \tag{5.4}$$

Now, setting $a = q$ and $b = q^5$ in (2.10), we deduce that

$$f(q, q^5) - f(-q, -q^5) = 2qf(q^4, q^{20}). \tag{5.5}$$

Employing (5.4) in (5.5), we find that

$$\sum_{n=0}^{\infty} asc_3(n)q^n - \sum_{n=0}^{\infty} (-1)^n asc_3(n)q^n = 2q \sum_{n=0}^{\infty} asc_3(n)q^{4n}. \tag{5.6}$$

Comparing coefficients of q^{4n+1} on both sides of (5.6), we immediately arrive at (5.1). □

Theorem 5.2. *If $asc_3(n)$ denotes the number of 3-cores of n that are self-conjugate, then*

$$asc_3(8P_k) = 1, \tag{5.7}$$

where P_k is either of the kth generalized pentagonal numbers $k(3k \pm 1)/2$.

Proof. Setting $a = q$ and $b = q^5$ in (2.9), we deduce that

$$f(q, q^5) + f(-q, -q^5) = 2f(q^8, q^{16}). \tag{5.8}$$

Now, from the definition (2.2) of $f(a, b)$, we see that

$$f(q, q^2) = \sum_{k=-\infty}^{\infty} q^{k(3k-1)/2}. \tag{5.9}$$

Employing (5.4) and (5.9) in (5.8), we find that

$$\sum_{n=0}^{\infty} asc_3(n)q^n + \sum_{n=0}^{\infty} (-1)^n asc_3(n)q^n = 2 \sum_{k=-\infty}^{\infty} q^{8k(3k-1)/2}. \tag{5.10}$$

Comparing coefficients of q^{8P_k} on both sides of (5.10), we readily deduce (5.7). □

The following identity can be found in a paper by F. Garvan, D. Kim, and D. Stanton [10, Corollary 2(1)].

Theorem 5.3. *If $asc_5(n)$ denotes the number of 5-cores of n that are self-conjugate, then*

$$asc_5(2n+1)=asc_5(n). \tag{5.11}$$

Proof. Setting $t=5$ in (2.26), we find that

$$\sum_{n=0}^{\infty} asc_5(n)q^n=\frac{\chi(q)f^2(-q^{10})}{\chi(q^5)}. \tag{5.12}$$

Using once again the Jacobi triple product identity (2.3), we find that

$$\begin{aligned} f(q,q^9)f(q^3,q^7)&=(-q;q^{10})_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(-q^9;q^{10})_\infty(q^{10};q^{10})_\infty^2\\ &=\frac{(-q;q^2)_\infty(q^{10};q^{10})_\infty^2}{(-q^5;q^{10})_\infty}=\frac{\chi(q)f^2(-q^{10})}{\chi(q^5)}. \end{aligned} \tag{5.13}$$

From (5.12) and (5.13), we can deduce that

$$\sum_{n=0}^{\infty} asc_5(n)q^n=f(q,q^9)f(q^3,q^7). \tag{5.14}$$

Now, setting $a=q$, $b=q^9$, $c=q^3$, and $d=q^7$ in (2.12), we deduce that

$$f(q,q^9)f(q^3,q^7)-f(-q,-q^9)f(-q^3,-q^7)=2qf(q^6,q^{14})f(q^2,q^{18}). \tag{5.15}$$

Employing (5.14) in (5.15), we find that

$$\sum_{n=0}^{\infty} asc_5(n)q^n-\sum_{n=0}^{\infty}(-1)^n asc_5(n)q^n=2q\sum_{n=0}^{\infty} asc_5(n)q^{2n}. \tag{5.16}$$

Comparing coefficients of q^{2n+1} on both sides of (5.16), we deduce (5.11). □

Theorem 5.4. *If $asc_5(n)$ denotes the number of 5-cores of n that are self-conjugate, and if $t_2(n)$ denotes the number of representations of n as a sum of two triangular numbers, then*

$$asc_5\big(4(5n+1)\big)=t_2(5n+1)-t_2(n). \tag{5.17}$$

Proof. Setting $a=q$, $b=q^9$, $c=q^3$, and $d=q^7$ in (2.11), we deduce that

$$f(q,q^9)f(q^3,q^7)+f(-q,-q^9)f(-q^3,-q^7)=2f(q^4,q^{16})f(q^8,q^{12}). \tag{5.18}$$

With the help of (5.14), we rewrite (5.18) as

$$\sum_{n=0}^{\infty} asc_5(n)q^n+\sum_{n=0}^{\infty}(-1)^n asc_5(n)q^n=2f(q^4,q^{16})f(q^8,q^{12}). \tag{5.19}$$

From the definition of $\psi(q)$ in (2.5), it is clear that the generating function for $t_2(n)$ is given by

$$\psi^2(q)=\sum_{n=0}^{\infty}t_2(n)q^n. \tag{5.20}$$

Thus, (2.14) can be rewritten in the form

$$f\left(q,q^4\right)f\left(q^2,q^3\right)=\sum_{n=0}^{\infty}t_2(n)q^n-\sum_{n=0}^{\infty}t_2(n)q^{5n+1}. \tag{5.21}$$

Replacing q by q^4 in (5.21), and then using this in (5.19), we find that

$$\sum_{n=0}^{\infty}asc_5(n)q^n+\sum_{n=0}^{\infty}(-1)^n asc_5(n)q^n=2\left(\sum_{n=0}^{\infty}t_2(n)q^{4n}-\sum_{n=0}^{\infty}t_2(n)q^{4(5n+1)}\right). \tag{5.22}$$

Comparing the coefficients of q^{20n+4} on both sides of (5.22), we easily deduce (5.17) to complete the proof. □

Remark. In a similar way, employing (2.13), we can easily show that

$$4asc_5(5n-1)=r_2(5n)-r_2(n),$$

where $r_2(n)$ denotes the number of representations of the positive integer n as a sum of two squares. This result was noticed earlier by Garvan, Kim, and Stanton [10, Eq. (7.3)].

6. New partition identities associated with modular equations of degree 3

Theorem 6.1. *Let S denote the set consisting of two copies, say in colors orange and blue, of the positive integers and one additional copy, say in color red, of those positive integers that are not multiples of 3. Let $A(N)$ and $B(N)$ denote the number of partitions of $2N$ into odd elements and even elements, respectively, of S. Then, for $N\geqslant 1$,*

$$A(N)=B(N).$$

Proof. From Entry 5(i) of Chapter 19 in Ramanujan's second notebook [13], [4, p. 230], we note that, if β has degree 3 over α, then

$$\left(\frac{\alpha^3}{\beta}\right)^{1/8}-\left(\frac{(1-\alpha)^3}{1-\beta}\right)^{1/8}=1. \tag{6.1}$$

Multiplying both sides of (6.1) by

$$\frac{(\beta(1-\beta))^{1/12}}{(\alpha(1-\alpha))^{1/8}},$$

we obtain the equivalent equation

$$\frac{\alpha^{1/4}}{(1-\alpha)^{1/8}}\cdot\frac{(1-\beta)^{1/12}}{\beta^{1/24}}-\frac{(1-\alpha)^{1/4}}{\alpha^{1/8}}\cdot\frac{\beta^{1/12}}{(1-\beta)^{1/24}}=\frac{(\beta(1-\beta))^{1/12}}{(\alpha(1-\alpha))^{1/8}}. \tag{6.2}$$

Transcribing (6.2) with the help of (2.21)–(2.23), we find that

$$2\frac{\chi(-q^3)}{\chi^3(-q^2)}-\frac{\chi^3(-q)}{\chi(-q^6)}=\frac{\chi^3(q)}{\chi^2(q^3)}. \tag{6.3}$$

Multiplying both sides of (6.3) by $\chi(q^3)$, we deduce that

$$\frac{\chi^3(q)}{\chi(q^3)}+\frac{\chi^3(-q)}{\chi(-q^3)}=2\frac{\chi(-q^6)}{\chi^3(-q^2)}. \tag{6.4}$$

Using (2.8) and (2.7), we can write (6.4) in the form

$$\frac{(-q;q^2)^3_\infty}{(-q^3;q^6)_\infty}+\frac{(q;q^2)^3_\infty}{(q^3;q^6)_\infty}=2\frac{(-q^2;q^2)^3_\infty}{(-q^6;q^6)_\infty}. \tag{6.5}$$

We can rewrite (6.5) in the form

$$\left(-q;q^2\right)^2_\infty\left(-q,-q^5;q^6\right)_\infty+\left(q;q^2\right)^2_\infty\left(q,q^5;q^6\right)_\infty=2\left(-q^2;q^2\right)^2_\infty\left(-q^2,-q^4;q^6\right)_\infty. \tag{6.6}$$

It is now readily seen that (6.6) is equivalent to Theorem 6.1. □

Theorem 6.2. *Let $A(N)$ and $B(N)$ be as defined in Theorem* 6.1. *Let $D(N)$ denote the number of partitions of $2N$ into parts congruent to* ± 2, ± 4, ± 6, ± 8, ± 10, ± 12, ± 14, *or* ± 16 *modulo* 36 *with parts congruent to* ± 2, ± 6, ± 10, *or* ± 14 *modulo* 36 *having one additional color, say green. Then, for $N \geqslant 1$,*

$$2A(N)=3D(N)=2B(N).$$

Proof. One of Ramanujan's modular equations of degree 3 can be written in the equivalent form [5, Entry 50(i), p. 202]

$$\frac{\chi^3(q)}{\chi(q^3)}=1+3q\frac{\psi(-q^9)}{\psi(-q)}. \tag{6.7}$$

Replacing q by $-q$ in (6.7), we obtain the equation

$$\frac{\chi^3(-q)}{\chi(-q^3)}=1-3q\frac{\psi(q^9)}{\psi(q)}. \tag{6.8}$$

Adding (6.7) and (6.8), we find that

$$\frac{\chi^3(q)}{\chi(q^3)}+\frac{\chi^3(-q)}{\chi(-q^3)}=2-3q\left\{\frac{\psi(q^9)}{\psi(q)}-\frac{\psi(-q^9)}{\psi(-q)}\right\}. \tag{6.9}$$

But, from Entry 4(i) in Chapter 20 of Ramanujan's second notebook [13], [4, p. 358],

$$q\left\{\frac{\psi(q^9)}{\psi(q)}-\frac{\psi(-q^9)}{\psi(-q)}\right\}=1-\frac{\phi(-q^{18})}{\phi(-q^2)}. \tag{6.10}$$

Employing (6.10) in (6.9), we find that

$$\frac{\chi^3(q)}{\chi(q^3)}+\frac{\chi^3(-q)}{\chi(-q^3)}=3\frac{\phi(-q^{18})}{\phi(-q^2)}-1. \tag{6.11}$$

Using (2.8), (2.4), and (6.6), we can deduce from (6.11) that

$$\begin{aligned}
&2\left(-q^2;q^2\right)_\infty^2\left(-q^2,-q^4;q^6\right)_\infty \\
&\quad=\left(-q;q^2\right)_\infty^2\left(-q,-q^5;q^6\right)_\infty+\left(q;q^2\right)_\infty^2\left(q,q^5;q^6\right)_\infty \\
&\quad=\frac{3}{(q^2,q^6,q^{10},q^{14},q^{22},q^{26},q^{30},q^{34};q^{36})_\infty^2(q^4,q^8,q^{12},q^{16},q^{20},q^{24},q^{28},q^{32};q^{36})_\infty} \\
&\qquad-1. \qquad (6.12)
\end{aligned}$$

We observe now that (6.12) is equivalent to the statement of Theorem 6.2. □

6.1. Example: $N=3$

Then $A(3)=12=B(3)$, with the twelve representations in odd and even elements being given respectively by

$$\begin{aligned}
&5_o+1_o=5_o+1_b=5_o+1_r=5_b+1_o=5_b+1_b=5_b+1_r=5_r+1_o=5_r+1_b \\
&\qquad=5_r+1_r=3_o+3_b=3_o+1_o+1_b+1_r=3_b+1_o+1_b+1_r, \\
&6_o=6_b=4_o+2_o=4_o+2_b=4_o+2_r=4_b+2_o=4_b+2_b=4_b+2_r \\
&\qquad=4_r+2_o=4_r+2_b=4_r+2_r=2_o+2_b+2_r.
\end{aligned}$$

Furthermore, $D(6)=8$, and the eight relative representations of 6 are given by

$$6=6_g=4+2=4+2_g=2+2+2=2+2+2_g=2+2_g+2_g=2_g+2_g+2_g.$$

Corollary 6.3. *We have $A(N)\equiv B(N)\equiv 0 \pmod 3$ and $D(N)\equiv 0 \pmod 2$.*

Theorem 6.4. *Let S denote the set consisting of one copy, say in color orange, of the positive integers and five additional copies, say in colors blue, red, green, pink, and violet, of those positive integers that are multiples of 3. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements of S, and let $B(N)$ denote the number of partitions of $2N$ into even elements of S. Furthermore, let $C(N)$ denote the number of partitions of N into 4 distinct colors, say Orange, Blue, Red, and Green, each appearing at most once and only in odd parts that are not multiples of 3. Then, for $N\geqslant 1$,*

$$C(2N)=A(N) \quad \text{and} \quad C(2N+1)=4B(N).$$

Proof. One of the many modular equations of degree 3 recorded by Ramanujan in his second notebook is given by [4, Entry 5(viii), p. 231]

$$\left(\alpha\beta^5\right)^{1/8}+\left\{(1-\alpha)(1-\beta)^5\right\}^{1/8}=1-\left(\frac{\beta^3(1-\alpha)^3}{\alpha(1-\beta)}\right)^{1/8}. \qquad (6.13)$$

Dividing both sides of (6.13) by $(\alpha(1-\alpha)\beta^5(1-\beta)^5)^{1/24}$, we obtain the equivalent modular equation

$$\begin{aligned}
&\frac{\alpha^{1/12}}{(1-\alpha)^{1/24}}\cdot\left(\frac{\beta^{1/12}}{(1-\beta)^{1/24}}\right)^5+\frac{(1-\alpha)^{1/12}}{\alpha^{1/24}}\cdot\left(\frac{(1-\beta)^{1/12}}{\beta^{1/24}}\right)^5 \\
&\quad=\frac{1}{(\alpha(1-\alpha)\beta^5(1-\beta)^5)^{1/24}}-\frac{(1-\alpha)^{1/3}}{\alpha^{1/6}}\cdot\frac{\beta^{1/6}}{(1-\beta)^{1/3}}. \qquad (6.14)
\end{aligned}$$

Transcribing (6.14) with the help of (2.21)–(2.23), we find that

$$\chi(q)\chi^5(q^3) - \chi(-q)\chi^5(-q^3) = \frac{8q^2}{\chi(-q^2)\chi^5(-q^6)} + 2q\frac{\chi^4(-q)}{\chi^4(-q^3)}. \tag{6.15}$$

Expressing (6.15) in q-products with the help of (2.8) and (2.7), we deduce that

$$\begin{aligned}(-q;q^2)_\infty(-q^3;q^6)_\infty^5 - (q;q^2)_\infty(q^3;q^6)_\infty^5 \\ = 8q^2(-q^2;q^2)_\infty(-q^6;q^6)_\infty^5 + 2q\frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4}.\end{aligned} \tag{6.16}$$

Replacing q by $-q$ in (6.16), we find that

$$\begin{aligned}(q;q^2)_\infty(q^3;q^6)_\infty^5 - (-q;q^2)_\infty(-q^3;q^6)_\infty^5 \\ = 8q^2(-q^2;q^2)_\infty(-q^6;q^6)_\infty^5 - 2q\frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4}.\end{aligned} \tag{6.17}$$

Subtracting (6.17) from (6.16), we deduce that

$$\begin{aligned}(-q;q^2)_\infty(-q^3;q^6)_\infty^5 - (q;q^2)_\infty(q^3;q^6)_\infty^5 \\ = q\left\{\frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4} + \frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4}\right\} = q\left\{(-q,-q^5;q^6)_\infty^4 + (q,q^5;q^6)_\infty^4\right\}.\end{aligned} \tag{6.18}$$

On the other hand, adding (6.16) and (6.17), we deduce that

$$\begin{aligned}8q(-q^2;q^2)_\infty(-q^6;q^6)_\infty^5 \\ = \frac{(-q;q^2)_\infty^4}{(-q^3;q^6)_\infty^4} - \frac{(q;q^2)_\infty^4}{(q^3;q^6)_\infty^4} = (-q,-q^5;q^6)_\infty^4 - (q,q^5;q^6)_\infty^4.\end{aligned} \tag{6.19}$$

It is now easy to see that (6.18) and (6.19) have the two partition-theoretic interpretations given in the statement of Theorem 6.4. □

6.2. *Example: $N = 3$*

Then $C(6) = 16 = A(3)$, and we have the representations

$$\begin{aligned}&5_O + 1_O = 5_O + 1_B = 5_O + 1_R = 5_O + 1_G, \quad \text{12 further partitions of the form } 5+1,\\ &7_o = 3_o + 3_b + 1_o, \quad \text{14 further partitions of the form } 3+3+1.\end{aligned}$$

Also, $C(7) = 28$ and $B(3) = 7$, which are evinced by the representations

$$\begin{aligned}&7_O = 7_B = 7_R = 7_G = 5_O + 1_O + 1_B = 5_O + 1_O + 1_R = 5_O + 1_O + 1_G\\ &\quad = 5_O + 1_B + 1_R = 5_O + 1_B + 1_G, \quad \text{19 additional representations of the form } 5+1+1,\\ &6_o = 6_b = 6_r = 6_g = 6_p = 6_v = 4_o + 2_o.\end{aligned}$$

7. New partition identities associated with modular equations of degree 5

Theorem 7.1. *Let S denote the set consisting of four copies, say in colors orange, blue, red, and green, of the positive integers and one additional copy, say in color pink, of those positive* *integers that are not multiples of* 5. *Let* $A(N)$ *and* $B(N)$ *denote the number of partitions of* $2N$ *into odd elements and even elements, respectively, of* S. *Then, for* $N \geqslant 1$,

$$A(N) = 2B(N).$$

Proof. If β has degree 5 over α, then [4, Entry 13(ii), p. 280]

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} - \left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = 1 + 2^{1/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{7.1}$$

Multiplying both sides of (7.1) by

$$\frac{(\beta(1-\beta))^{1/12}}{(\alpha(1-\alpha))^{5/24}}$$

we arrive at the equivalent modular equation

$$\begin{aligned}&\left(\frac{\alpha^{1/12}}{(1-\alpha)^{1/24}}\right)^5 \cdot \frac{(1-\beta)^{1/12}}{\beta^{1/24}} - \left(\frac{(1-\alpha)^{1/12}}{\alpha^{1/24}}\right)^5 \cdot \frac{\beta^{1/12}}{(1-\beta)^{1/24}} \\ &\quad = \frac{(\beta(1-\beta))^{1/12}}{(\alpha(1-\alpha))^{5/24}} + 2^{1/3}\big(\beta(1-\beta)\big)^{1/24}.\end{aligned} \tag{7.2}$$

Transcribing (7.2) with the help of (2.21)–(2.23), we find that

$$4\frac{\chi(-q^5)}{\chi^5(-q^2)} - \frac{\chi^5(-q)}{\chi(-q^{10})} = \frac{\chi^5(q)}{\chi^2(q^5)} + \frac{2}{\chi(q^5)}. \tag{7.3}$$

Multiplying both sides of (7.3) by $\chi(q^5)$, we deduce that

$$\frac{\chi^5(q)}{\chi(q^5)} + \frac{\chi^5(-q)}{\chi(-q^5)} + 2 = 4\frac{\chi(-q^{10})}{\chi^5(-q^2)}. \tag{7.4}$$

Employing (2.8) and (2.7), we can rewrite (7.4) in the form

$$\frac{(-q;q^2)_\infty^5}{(-q^5;q^{10})_\infty} + \frac{(q;q^2)_\infty^5}{(q^5;q^{10})_\infty} + 2 = 4\frac{(-q^2;q^2)_\infty^5}{(-q^{10};q^{10})_\infty}, \tag{7.5}$$

which we can rewrite in the form

$$\begin{aligned}&\big(-q;q^2\big)_\infty^4\big(-q,-q^3,-q^7,-q^9;q^{10}\big)_\infty + \big(q;q^2\big)_\infty^4\big(q,q^3,q^7,q^9;q^{10}\big)_\infty + 2 \\ &\quad = 4\big(-q^2;q^2\big)_\infty^4\big(-q^2,-q^4,-q^6,-q^8;q^{10}\big)_\infty.\end{aligned} \tag{7.6}$$

It is now readily seen that (7.6) has the partition-theoretic interpretation claimed in Theorem 7.1. □

7.1. Example: $N = 3$

Then $A(3) = 80$, $B(3) = 40$, and we record the representations

$$\begin{aligned}5_o + 1_o &= 5_o + 1_b, \quad \text{18 further representations of the form } 5+1, \\ &= 3_o + 3_b = 3_o + 3_r \quad \text{8 further representations of the form } 3+3, \\ &= 3_o + 1_o + 1_b + 1_r, \quad \text{49 additional representations of the form } 3+1+1+1.\end{aligned}$$

$$\begin{aligned}6_o &= 6_b = 6_r = 6_g = 6_p = 4_o + 2_o = 4_o + 2_b = 4_o + 2_r = 4_o + 2_g\\ &= 4_o + 2_p, \quad \text{20 additional representations of the form } 4+2,\\ &= 2_o + 2_b + 2_r, \quad \text{9 additional representations of the form } 2+2+2.\end{aligned}$$

Theorem 7.2. *Let $A(N)$ and $B(N)$ be as defined in Theorem* 6.1. *Let $D(N)$ denote the number of partitions of $2N$ into parts congruent to ± 2, ± 4, ± 6, or ± 8 modulo* 20 *having two colors, say Orange and Blue, with parts congruent to ± 2 or ± 6 modulo* 20 *having two additional colors, say Red and Green. Then, for $N \geqslant 1$,*

$$2A(N) = 5D(N) = 4B(N).$$

Proof. Another of Ramanujan's modular equations of degree 5 is given by [5, p. 202]

$$\frac{\chi^5(q)}{\chi(q^5)} = 1 + 5q\frac{\psi^2(-q^5)}{\psi^2(-q)}. \tag{7.7}$$

Replacing q by $-q$ in (7.7) gives

$$\frac{\chi^5(-q)}{\chi(-q^5)} = 1 - 5q\frac{\psi^2(q^5)}{\psi^2(q)}. \tag{7.8}$$

Adding (7.7) and (7.8), we find that

$$\frac{\chi^5(q)}{\chi(q^5)} + \frac{\chi^5(-q)}{\chi(-q^5)} = 2 - 5q\left\{\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right\}. \tag{7.9}$$

Arising in the proof of another of Ramanujan's modular equations of degree 5 is the identity [4, Eq. (12.32), p. 276]

$$q\left\{\frac{\psi^2(q^5)}{\psi^2(q)} - \frac{\psi^2(-q^5)}{\psi^2(-q)}\right\} = 1 - \frac{\phi^2(-q^{10})}{\phi^2(-q^2)}. \tag{7.10}$$

Employing (7.10) in (7.9), we find that

$$\frac{\chi^5(q)}{\chi(q^5)} + \frac{\chi^5(-q)}{\chi(-q^5)} = 5\frac{\phi^2(-q^{10})}{\phi^2(-q^2)} - 3. \tag{7.11}$$

Using (2.8), (2.4), and (7.6) in (7.11), we conclude that

$$\begin{aligned}&4\big(-q^2;q^2\big)_\infty^4\big(-q^2,-q^4,-q^6,-q^8;q^{10}\big)_\infty - 2\\ &\quad= \big(-q;q^2\big)_\infty^4\big(-q,-q^3,-q^7,-q^9;q^{10}\big)_\infty + \big(q;q^2\big)_\infty^4\big(q,q^3,q^7,q^9;q^{10}\big)_\infty\\ &\quad= \frac{5}{(q^2,q^6,q^{14},q^{18};q^{20})_\infty^4(q^4,q^8,q^{12},q^{16};q^{20})_\infty^2} - 3.\end{aligned} \tag{7.12}$$

The partition-theoretic interpretation of (7.12) in Theorem 7.2 now follows easily. □

7.2. *Example: $N = 3$*

We have already observed in the previous example that $A(3) = 80$ and $B(3) = 40$. We see that $D(3) = 32$, and the relevant representations are given by

$$\begin{aligned}6_O &= 6_B = 6_R = 6_G = 4_O + 2_O, \quad \text{7 further representations of the form } 4+2,\\ &= 2_O + 2_O + 2_O, \quad \text{19 further representations of the form } 2+2+2.\end{aligned}$$

Corollary 7.3. *We have* $A(N)$, $B(N) \equiv 0 \pmod 5$ *and* $D(N) \equiv 0 \pmod 4$.

Theorem 7.4. *Let S denote the set consisting of one copy, say in color orange, of the positive integers and three additional copies, say in colors blue, red, and green, of those positive integers that are multiples of 5. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements of S, and let $B(N)$ denote the number of partitions of $2N$ into even elements of S. Furthermore, let $C(N)$ denote the number of partitions of N into 2 distinct colors, say Orange and Blue, each appearing at most once and only in odd parts that are not multiples of 5. Then, for $N \geqslant 1$,*

$$C(2N) = A(N) \quad \textit{and} \quad C(2N+1) = 2B(N).$$

Proof. Recording another of Ramanujan's modular equations of degree 5 from his notebooks [4, Entry 13(vii), p. 281], we have

$$\left(\alpha\beta^3\right)^{1/8} + \left\{(1-\alpha)(1-\beta)^3\right\}^{1/8} = 1 - 2^{1/3}\left(\frac{\beta^5(1-\alpha)^5}{\alpha(1-\beta)}\right)^{1/24}. \tag{7.13}$$

Dividing both sides of (7.13) by $(\alpha(1-\alpha)\beta^3(1-\beta)^3)^{1/24}$, we obtain the alternative formulation

$$\begin{aligned} &\frac{\alpha^{1/12}}{(1-\alpha)^{1/24}} \cdot \left(\frac{\beta^{1/12}}{(1-\beta)^{1/24}}\right)^3 + \frac{(1-\alpha)^{1/12}}{\alpha^{1/24}} \cdot \left(\frac{(1-\beta)^{1/12}}{\beta^{1/24}}\right)^3 \\ &\quad = \frac{1}{(\alpha(1-\alpha)\beta^3(1-\beta)^3)^{1/24}} - 2^{1/3}\frac{(1-\alpha)^{1/6}}{\alpha^{1/12}} \cdot \frac{\beta^{1/12}}{(1-\beta)^{1/6}}. \end{aligned} \tag{7.14}$$

Transcribing (7.14) with the help of (2.21)–(2.23), we find that

$$\chi(q)\chi^3\left(q^5\right) - \chi(-q)\chi^3\left(-q^5\right) = \frac{4q^2}{\chi(-q^2)\chi^3(-q^{10})} + 2q\frac{\chi^2(-q)}{\chi^2(-q^5)}. \tag{7.15}$$

Expressing (7.15) in terms of q-products with the help of (2.8) and (2.7), we deduce that

$$\begin{aligned} &\left(-q;q^2\right)_\infty\left(-q^5;q^{10}\right)_\infty^3 - \left(q;q^2\right)_\infty\left(q^5;q^{10}\right)_\infty^3 \\ &\quad = 4q^2\left(-q^2;q^2\right)_\infty\left(-q^{10};q^{10}\right)_\infty^3 + 2q\frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}. \end{aligned} \tag{7.16}$$

Replacing q by $-q$ in (7.16), we find that

$$\begin{aligned} &\left(q;q^2\right)_\infty\left(q^5;q^{10}\right)_\infty^3 - \left(-q;q^2\right)_\infty\left(-q^5;q^{10}\right)_\infty^3 \\ &\quad = 4q^2\left(-q^2;q^2\right)_\infty\left(-q^{10};q^{10}\right)_\infty^3 - 2q\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2}. \end{aligned} \tag{7.17}$$

Subtracting (7.17) from (7.16), we deduce that

$$\begin{aligned} &\left(-q;q^2\right)_\infty\left(-q^5;q^{10}\right)_\infty^3 - \left(q;q^2\right)_\infty\left(q^5;q^{10}\right)_\infty^3 \\ &\quad = q\left\{\frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2} + \frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}\right\} \\ &\quad = q\left\{\left(-q,-q^3,-q^7,-q^9;q^{10}\right)_\infty^2 + \left(q,q^3,q^7,q^9;q^{10}\right)_\infty^2\right\}. \end{aligned} \tag{7.18}$$

On the other hand, adding (7.16) and (7.17), we deduce that

$$4q(-q^2;q^2)_\infty(-q^{10};q^{10})_\infty^3 = \frac{(-q;q^2)_\infty^2}{(-q^5;q^{10})_\infty^2} - \frac{(q;q^2)_\infty^2}{(q^5;q^{10})_\infty^2}$$
$$= (-q,-q^3,-q^7,-q^9;q^{10})_\infty^2 - (q,q^3,q^7,q^9;q^{10})_\infty^2. \tag{7.19}$$

It is now easy to see that (7.18) and (7.19) have the partition-theoretic interpretations claimed in Theorem 7.4. □

7.3. Example: $N = 6$

Then $C(12) = 12 = A(6)$, and we have the representations

$$\begin{aligned} 11_O + 1_O &= 11_O + 1_B = 11_B + 1_O = 11_B + 1_B = 9_O + 3_O = 9_O + 3_B = 9_B + 3_O \\ &= 9_B + 3_B = 7_O + 3_O + 1_O + 1_B = 7_O + 3_B + 1_O + 1_B \\ &= 7_B + 3_O + 1_O + 1_B = 7_B + 3_B + 1_O + 1_B, \end{aligned}$$

$$\begin{aligned} 13_o &= 9_o + 3_o + 1_o = 7_o + 5_o + 1_o = 7_o + 5_b + 1_o = 7_o + 5_r + 1_o = 7_o + 5_g + 1_o \\ &= 5_o + 5_b + 3_o = 5_o + 5_r + 3_o = 5_o + 5_g + 3_o = 5_b + 5_r + 3_o \\ &= 5_b + 5_g + 3_o = 5_r + 5_g + 3_o. \end{aligned}$$

Also, $C(13) = 14$ and $B(6) = 7$, and we have the representations

$$\begin{aligned} 13_O &= 13_B = 11_O + 1_O + 1_B = 11_B + 1_O + 1_B = 9_O + 3_O + 1_O = 9_O + 3_O + 1_B \\ &= 9_O + 3_B + 1_O = 9_O + 3_B + 1_B = 9_B + 3_O + 1_O = 9_B + 3_O + 1_B \\ &= 9_B + 3_B + 1_O = 9_B + 3_B + 1_B = 7_O + 3_O + 3_B = 7_B + 3_O + 3_B, \end{aligned}$$

$$12_o = 10_o + 2_o = 10_b + 2_o = 10_r + 2_o = 10_g + 2_o = 8_o + 4_o = 6_o + 4_o + 2_o.$$

8. New partition identities associated with modular equations of degree 15

Theorem 8.1. *Let S denote the set consisting of one copy, say in color orange, of the positive integers and three additional copies, say in colors blue, green, and red, of those positive integers that are multiples of* 3, 5, *and* 15, *respectively. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements of S, and let $B(N)$ denote the number of partitions of $2N-2$ into even elements of S, with the convention that $B(1) = 1$. Then $A(0) = 1$, and for $N \geqslant 1$,*

$$A(N) = 2B(N).$$

Proof. We begin with a modular equation of degree 15 recorded first by Weber [20] and later by Ramanujan in his second notebook [4, Entry 11(xiv), p. 385]. If β, γ, and δ have degrees 3, 5, and 15, respectively, over α, then

$$(\alpha\beta\gamma\delta)^{1/8} + \{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/8}$$
$$+ 2^{1/3}\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24} = 1. \tag{8.1}$$

Multiplying both sides of (8.1) by $2^{2/3}q/\{\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{1/24}$, we find that

$$\begin{aligned}
&\frac{4q^3\alpha^{1/12}}{2^{1/3}q^{1/12}(1-\alpha)^{1/24}}\cdot\frac{\beta^{1/12}}{2^{1/3}q^{3/12}(1-\beta)^{1/24}}\\
&\quad\cdot\frac{\gamma^{1/12}}{2^{1/3}q^{5/12}(1-\gamma)^{1/24}}\cdot\frac{\delta^{1/12}}{2^{1/3}q^{15/12}(1-\delta)^{1/24}}\\
&\quad+\frac{2^{1/6}q^{1/24}(1-\alpha)^{1/12}}{\alpha^{1/24}}\cdot\frac{2^{1/6}q^{3/24}(1-\beta)^{1/12}}{\beta^{1/24}}\\
&\quad\cdot\frac{2^{1/6}q^{5/24}(1-\gamma)^{1/12}}{\gamma^{1/24}}\cdot\frac{2^{1/6}q^{15/24}(1-\delta)^{1/12}}{\delta^{1/24}}+2q\\
&\qquad=\frac{2^{1/6}q^{1/24}}{\{\alpha(1-\alpha)\}^{1/24}}\cdot\frac{2^{1/6}q^{3/24}}{\{\beta(1-\beta)\}^{1/24}}\cdot\frac{2^{1/6}q^{5/24}}{\{\gamma(1-\gamma)\}^{1/24}}\cdot\frac{2^{1/6}q^{15/24}}{\{\delta(1-\delta)\}^{1/24}}. \qquad (8.2)
\end{aligned}$$

Converting (8.2) with the help of (2.21)–(2.23), we find that

$$\begin{aligned}
&\frac{4q^3}{\chi(-q^2)\chi(-q^6)\chi(-q^{10})\chi(-q^{30})}+\chi(-q)\chi(-q^3)\chi(-q^5)\chi(-q^{15})+2q\\
&\quad=\chi(q)\chi(q^3)\chi(q^5)\chi(q^{15}). \qquad (8.3)
\end{aligned}$$

Employing (2.8) and (2.7) in (8.3), we deduce that

$$\begin{aligned}
&(-q;q^2)_\infty(-q^3;q^6)_\infty(-q^5;q^{10})_\infty(-q^{15};q^{30})_\infty\\
&\quad-(q;q^2)_\infty(q^3;q^6)_\infty(q^5;q^{10})_\infty(q^{15};q^{30})_\infty\\
&\qquad=2q+4q^3(-q^2;q^2)_\infty(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty(-q^{30};q^{30})_\infty. \qquad (8.4)
\end{aligned}$$

It is now easy to see that (8.4) is equivalent to the statement in Theorem 8.1. □

8.1. Example: $N=7$

Then $A(7)=16$ and $B(7)=8$, and the relevant partitions are given by

$$\begin{aligned}
15_o&=15_b=15_g=15_r=11_o+3_o+1_o=11_o+3_b+1_o=9_o+5_o+1_o=9_o+5_g+1_o\\
&=9_b+5_o+1_o=9_b+5_g+1_o=9_o+3_o+3_b=9_b+3_o+3_b\\
&=7_o+5_o+3_o=7_o+5_g+3_o=7_o+5_o+3_b=7_o+5_g+3_b,\\
12_o&=12_b=10_o+2_o=10_g+2_o=8_o+4_o=6_o+6_b=6_o+4_o+2_o=6_b+4_o+2_o.
\end{aligned}$$

Theorem 8.2. *Let S denote the set consisting of one copy, say in color orange, of those positive integers that are multiples of 3 and another copy, say in color blue, of those positive integers that are multiples of 5. Let $A(N)$ and $B(N)$ denote the number of partitions of $2N$ into, respectively, odd elements of S and even elements of S. Furthermore, let $C(N)$ denote the number of partitions of N into distinct odd parts that are not multiples of 3 or 5. Then, for $N\geqslant 6$,*

$$C(2N)=A(N)\quad\text{and}\quad C(2N+1)=B(N).$$

Proof. By Entry 11(iv) in Chapter 20 of Ramanujan's second notebook [13], [4, p. 383],

$$1+(\beta\gamma)^{1/8}+\{(1-\beta)(1-\gamma)\}^{1/8}=2^{2/3}\left(\frac{\beta^2\gamma^2(1-\beta)^2(1-\gamma)^2}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24}, \qquad (8.5)$$

where β, γ, and δ have degrees 3, 5, and 15, respectively, over α. Dividing both sides of (8.5) by $(\beta\gamma(1-\beta)(1-\gamma))^{1/24}$, we obtain the equivalent equation

$$\frac{\beta^{1/12}}{(1-\beta)^{1/24}}\cdot\frac{\gamma^{1/12}}{(1-\gamma)^{1/24}}+\frac{(1-\beta)^{1/12}}{\beta^{1/24}}\cdot\frac{(1-\gamma)^{1/12}}{\gamma^{1/24}}+\frac{1}{(\beta\gamma(1-\beta)(1-\gamma))^{1/24}}$$
$$=2^{2/3}\left(\frac{\beta\gamma(1-\beta)(1-\gamma)}{\alpha\delta(1-\alpha)(1-\delta)}\right)^{1/24}. \tag{8.6}$$

Transcribing (8.6) with the help of (2.21)–(2.23), we find that

$$\chi(q^3)\chi(q^5)+\chi(-q^3)\chi(-q^5)+\frac{2q}{\chi(-q^6)\chi(-q^{10})}=2\frac{\chi(q)\chi(q^{15})}{\chi(q^3)\chi(q^5)}. \tag{8.7}$$

Writing (8.7) in q-products with the help of (2.8) and (2.7), we deduce that

$$\begin{aligned}&(-q^3;q^6)_\infty(-q^5;q^{10})_\infty+(q^3;q^6)_\infty(q^5;q^{10})_\infty+2q(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty\\&\quad=2\frac{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}\\&\quad=2(-q,-q^7,-q^{11},-q^{13},-q^{17},-q^{19},-q^{23},-q^{29};q^{30})_\infty.\end{aligned} \tag{8.8}$$

Replacing q by $-q$ in (8.8) gives

$$\begin{aligned}&(-q^3;q^6)_\infty(-q^5;q^{10})_\infty+(q^3;q^6)_\infty(q^5;q^{10})_\infty-2q(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty\\&\quad=2(q,q^7,q^{11},q^{13},q^{17},q^{19},q^{23},q^{29};q^{30})_\infty.\end{aligned} \tag{8.9}$$

Adding (8.8) and (8.9), we deduce that

$$\begin{aligned}&(-q^3;q^6)_\infty(-q^5;q^{10})_\infty+(q^3;q^6)_\infty(q^5;q^{10})_\infty\\&\quad=(-q,-q^7,-q^{11},-q^{13},-q^{17},-q^{19},-q^{23},-q^{29};q^{30})_\infty\\&\qquad+(q,q^7,q^{11},q^{13},q^{17},q^{19},q^{23},q^{29};q^{30})_\infty.\end{aligned} \tag{8.10}$$

On the other hand, subtracting (8.9) from (8.8), we find that

$$\begin{aligned}2q(-q^6;q^6)_\infty(-q^{10};q^{10})_\infty&=(-q,-q^7,-q^{11},-q^{13},-q^{17},-q^{19},-q^{23},-q^{29};q^{30})_\infty\\&\quad-(q,q^7,q^{11},q^{13},q^{17},q^{19},q^{23},q^{29};q^{30})_\infty.\end{aligned} \tag{8.11}$$

It is now easy to see that (8.10) and (8.11) have the partition-theoretic interpretations claimed in our theorem. □

8.2. *Example: $N=10$*

Then $C(20)=2=A(10)$, and the relevant representations are given by

$$19+1=13+7,\qquad 15_o+5_b=15_b+5_b.$$

Next, $C(21)=1=B(10)$, and the single representations are

$$13+7+1,\qquad 20_b.$$

Theorem 8.3. *Let S denote the set consisting of one copy, say in color orange, of the positive integers and one additional copy, say in color blue, of those positive integers that are multiples of 15. Let $A(N)$ denote the number of partitions of $2N+1$ into odd elements of S, and let $B(N)$ denote the number of partitions of $2N$ into even elements of S. Furthermore, let $C(N)$ denote the number of partitions of N into odd parts that are not multiples of 3 or 5. Then, for $N \geqslant 6$,*

$$C(2N) = A(N) \quad \text{and} \quad C(2N+1) = B(N).$$

Proof. From Entry 11(v) in Chapter 20 of Ramanujan's second notebook [13], [4, p. 384],

$$1-(\alpha\delta)^{1/8}-\left\{(1-\alpha)(1-\delta)\right\}^{1/8}=2^{2/3}\left(\frac{\alpha^2\delta^2(1-\alpha)^2(1-\delta)^2}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}, \tag{8.12}$$

where β, γ, and δ have degrees 3, 5, and 15, respectively, over α. Dividing both sides of (8.12) by $(\alpha\delta(1-\alpha)(1-\delta))^{1/24}$, we obtain the equality

$$\begin{aligned}&\frac{1}{(\alpha\delta(1-\alpha)(1-\delta))^{1/24}}-\frac{\alpha^{1/12}}{(1-\alpha)^{1/24}}\cdot\frac{\delta^{1/12}}{(1-\delta)^{1/24}}-\frac{(1-\alpha)^{1/12}}{\alpha^{1/24}}\cdot\frac{(1-\delta)^{1/12}}{\delta^{1/24}}\\&\quad=2^{2/3}\left(\frac{\alpha\delta(1-\alpha)(1-\delta)}{\beta\gamma(1-\beta)(1-\gamma)}\right)^{1/24}.\end{aligned} \tag{8.13}$$

Transcribing (8.13) with the help of (2.21)–(2.23), we find that

$$\chi(q)\chi\left(q^{15}\right)-\chi(-q)\chi\left(-q^{15}\right)-\frac{2q^2}{\chi(-q^2)\chi(-q^{30})}=2q\frac{\chi(q^3)\chi(q^5)}{\chi(q)\chi(q^{15})}. \tag{8.14}$$

Writing (8.14) in q-products with the help of (2.8) and (2.7), we deduce that

$$\begin{aligned}&\left(-q;q^2\right)_\infty\left(-q^{15};q^{30}\right)_\infty-\left(q;q^2\right)_\infty\left(q^{15};q^{30}\right)_\infty-2q^2\left(-q^2;q^2\right)_\infty\left(-q^{30};q^{30}\right)_\infty\\&\quad=2q\frac{(-q^3;q^6)_\infty(-q^5;q^{10})_\infty}{(-q;q^2)_\infty(-q^{15};q^{30})_\infty}\\&\quad=\frac{2q}{(-q,-q^7,-q^{11},-q^{13},-q^{17},-q^{19},-q^{23},-q^{29};q^{30})_\infty}.\end{aligned} \tag{8.15}$$

Replacing q by $-q$ in (8.15), we find that

$$\begin{aligned}&\left(q;q^2\right)_\infty\left(q^{15};q^{30}\right)_\infty-\left(-q;q^2\right)_\infty\left(-q^{15};q^{30}\right)_\infty-2q^2\left(-q^2;q^2\right)_\infty\left(-q^{30};q^{30}\right)_\infty\\&\quad=-\frac{2q}{(q,q^7,q^{11},q^{13},q^{17},q^{19},q^{23},q^{29};q^{30})_\infty}.\end{aligned} \tag{8.16}$$

Subtracting (8.16) from (8.15), we find that

$$\begin{aligned}&\left(-q;q^2\right)_\infty\left(-q^{15};q^{30}\right)_\infty-\left(q;q^2\right)_\infty\left(q^{15};q^{30}\right)_\infty\\&\quad=q\left\{\frac{1}{(q,q^7,q^{11},q^{13},q^{17},q^{19},q^{23},q^{29};q^{30})_\infty}\right.\\&\qquad\left.+\frac{1}{(-q,-q^7,-q^{11},-q^{13},-q^{17},-q^{19},-q^{23},-q^{29};q^{30})_\infty}\right\}.\end{aligned} \tag{8.17}$$

On the other hand, adding (8.15) and (8.16), we deduce that

$$\begin{aligned}&2q(-q^2;q^2)_\infty(-q^{30};q^{30})_\infty\\&\quad=\frac{1}{(q,q^7,q^{11},q^{13},q^{17},q^{19},q^{23},q^{29};q^{30})_\infty}\\&\qquad-\frac{1}{(-q,-q^7,-q^{11},-q^{13},-q^{17},-q^{19},-q^{23},-q^{29};q^{30})_\infty}.\end{aligned}\tag{8.18}$$

Interpreting (8.17) and (8.18) in terms of partitions, we readily deduce the two respective partition identities of Theorem 8.3. □

8.3. Example: $N=6$

Then $C(12)=3=A(6)$, with the relevant representations being

$$\begin{aligned}&13_o=9_o+3_o+1_o=7_o+5_o+1_o,\\&11+1=7+1+1+1+1+1=1+1+1+1+1+1+1+1+1+1+1+1.\end{aligned}$$

Furthermore, $C(13)=4=B(6)$, and the representations that illustrate our theorem are given by

$$\begin{aligned}&12_o=10_o+2_o=8_o+4_o=6_o+4_o+2_o,\\&13=11+1+1=7+1+1+1+1+1+1\\&\quad=1+1+1+1+1+1+1+1+1+1+1+1+1.\end{aligned}$$

Note added in proof

K. Ono has kindly informed us that Theorems 4.1 and 4.2 follow from results on pages 10 and 9, respectively, of his paper with A. Granville [11].

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References

[1] C. Adiga, B.C. Berndt, S. Bhargava, G.N. Watson, Chapter 16 of Ramanujan's second notebook: Theta-functions and q-series, Mem. Amer. Math. Soc. 53 (315) (1985).
[2] G.E. Andrews, B.C. Berndt, Ramanujan's Lost Notebook, Part I, Springer, New York, 2005.
[3] N.D. Baruah, B.C. Berndt, Partition identities arising from theta function identities, submitted for publication.
[4] B.C. Berndt, Ramanujan's Notebooks, Part III, Springer-Verlag, New York, 1991.
[5] B.C. Berndt, Ramanujan's Notebooks, Part IV, Springer-Verlag, New York, 1994.
[6] B.C. Berndt, Ramanujan's Notebooks, Part V, Springer-Verlag, New York, 1998.
[7] B.C. Berndt, Partition-theoretic interpretations of certain modular equations of Schröter, Russell, and Ramanujan, Ann. Comb., in press.
[8] H.M. Farkas, I. Kra, Partitions and theta constant identities, in: The Mathematics of Leon Ehrenpreis, in: Contemp. Math., vol. 251, Amer. Math. Soc., Providence, RI, 2000, pp. 197–203.
[9] H.M. Farkas, I. Kra, Theta Constants, Riemann Surfaces and the Modular Group, Grad. Stud. Math., vol. 37, Amer. Math. Soc., Providence, RI, 2001.
[10] F. Garvan, D. Kim, D. Stanton, Cranks and t-cores, Invent. Math. 101 (1990) 1–17.
[11] A. Granville, K. Ono, Defect zero p-blocks for finite simple groups, Trans. Amer. Math. Soc. 348 (1996) 331–347.
[12] M.D. Hirschhorn, The case of the mysterious sevens, Int. J. Number Theory 2 (2006) 213–216.

[13] S. Ramanujan, Notebooks (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.
[14] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.
[15] H. Schröter, De aequationibus modularibus, Dissertatio Inauguralis, Albertina Litterarum Universitate, Regiomonti, 1854.
[16] H. Schröter, Über Modulargleichungen der elliptischen Funktionen, Auszug aus einem Schreiben an Herrn L. Kronecker, J. Reine Angew. Math. 58 (1861) 378–379.
[17] H. Schröter, Beiträge zur Theorie der elliptischen Funktionen, Acta Math. 5 (1884) 205–208.
[18] J.A. Sellers, An elementary proof of an infinite family of identities for 3-cores, preprint, May 10, 2006.
[19] S.O. Warnaar, A generalization of the Farkas and Kra partition theorem for modulus 7, J. Combin. Theory Ser. A 110 (2005) 43–52.
[20] H. Weber, Zur Theorie der elliptischen Funktionen, Acta Math. 11 (1887) 333–390.